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Numerical Integral Equation Method for ARL of CUSUM Chart for Long-Memory Process with Non-Seasonal and Seasonal ARFIMA Models

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Abstract

This paper presents an approximate average run length (ARL) of CUSUM chart for long-memory process by using numerical integral equation (NIE) method based on Gauss-Legendre quadrature rule. Measurement was a performance with the ARL between NIE method and explicit formulas in terms of percentage error when observations are non-seasonal and seasonal ARFIMA models with exponential white noise. Results indicated that ARL values by using both methods were similar and excellent agreement with the percentage error. Apparently, the NIE method is an alternative to explicit formulas for two models.

Keywords: Average Run Length (ARL), cumulative sum (CUSUM) chart, exponential white noise, Numerical Integral Equation (NIE) method, ARFIMA(p,d,q) process, SARFIMA(P,D,Q)$_S$ process.

1. Introduction

Statistical Process Control (SPC) plays an essential role in quality management and has been applied in many fields such as sciences, economics, engineering, finance, and medicine. However, the most applied tools in SPC are used in control chart; one which should be mentioned is the "Cumulative Sum (CUSUM) control chart". CUSUM chart has been widely used to monitor the detection of small process shifts and control the quality of products from manufacturing processes. A review of CUSUM chart was initially proposed by Page (1954) and has been studied in many literatures; particularly, see Gan (1991), Luceno and Puig-Pey (2006), Wu and Wang (2007). An extensively used measurement for CUSUM chart involves evaluation and comparing the performance of the control chart based on the average run length (ARL). The ARL measures the average number of observations taken before the signals. The in-control ARL (abbreviation ARL_0) is ordinarily fixed and referred to the average number of observations from the in-control process before a false out-of-control alarm is raised, which is a measurement of the false-alarm rate. On the contrary, out-of-control ARL (abbreviation ARL_1), which is the average number of observations required to detect a specific mean shift, represents the detection power of the control chart (Ryu et al. 2010). The performance of

CUSUM chart in the presence of autocorrelation has been studied in a number of aspects. See, for example, Johnson and Bagshaw (1974), Lu and Reynolds (2001), and Kim et al. (2007).

The class of autoregressive fractionally integrated moving average model can be denoted by ARFIMA (or non-seasonal ARFIMA) model and seasonal ARFIMA denote by SARFIMA model. These models were proposed by Granger and Joyeux (1980) and Hosking (1981). The SARFIMA model is a straightforward extension of the ARFIMA model. A fractionally integrated variant of ARMA and SARMA model can be also be defined by allowing both the non-seasonal and seasonal differencing parameters d and D respectively, to take non-integer valued Ray (1993), and use frequently to model a long-memory process, a more detailed description can be found in, e.g., Beran (1994), Ooms (1995), Baillie (1996), Palma (2007), Bisognin and Lopes (2007). The long-memory process is involved in a number of applications including finance and economics, environmental, sciences, and engineering. A comparison between seasonal ARFIMA models and standard (non-fractional) seasonal ARIMA models were presented by Ray (1993). The results showed that higher order AR models are capable of forecasting the longer term well when compared with ARFIMA models.

There are many control charts connected with time series following the ARFIMA model. For instance, Ramjee (2000) also analyzed the performance of Shewhart and EWMA charts for the presence of correlated data which occurred from an ARFIMA model. The studied result showed that these charts cannot perform well because of detecting process shifts. And thus, a new type of control chart and Hyperbolic Weighted Moving Average (HWMA) control chart was proposed. Two years later, Ramjee et al. (2002) presented a HWMA chart forecast-based control chart, specially designed for non-stationary ARFIMA models. Control chart for autocorrelated data using ARFIMA and ARIMA models are utilized with application to monitor air quality's data in Taiwan. As a result, the residual control charts using an ARFIMA model were more appropriate than ARIMA model was the conclusion in a study by Pan and Chen (2008). Recently, Rabyk and Schmid (2016) introduced the EWMA charts to detect changes in the mean of a long-memory process. The designed control chart was calculated from an ARFIMA(p,d,q) process.

Exponential white noise coordinated with time series has also been examined completely. The autoregressive moving average process order (1,1) denoted by ARMA(1,1) when observations are exponentially distributed with exponential white noise was considered by Jacob and Lewis (1977). Later, Mohamed and Hocine (2003) applied Bayesian analysis of the autoregressive model order 1 denoted by AR(1) with an exponential distribution six years later. Moreover, Pereira and Turkrman (2004) used exponential white noise to develop a Bayesian analysis of a threshold autoregressive model.

In related literatures, there are several methods that can be utilized to find the average run length (ARL) such as Monte Carlo simulations (MC), Markov chain approach (MCA), numerical integral equation (NIE) method and explicit formulas. For example, Vanbrackle and Reynold (1997) studied EWMA and CUSUM charts by using an integral equation and Markov chain approach to evaluate the ARL in the case of AR(1) process with additional random error. Sukparungsee and Novikov (2006) used the Martingale approach to derive the analytical formulas of the average run length (ARL) and the average delay (AD) in the case of Gaussian and a few non-Gaussian distributions. Later, Areepong and Novikov (2009) derived the explicit formulas of average run length (ARL) for EWMA chart with exponential distribution. The explicit formulas of ARL was recently presented by Mititelu et al. (2010) who used Fredholm integral equation for one-sided EWMA chart for the case of laplace distribution and CUSUM chart with hyperexponential distribution. The numerical integral equation (NIE) method of ARL for CUSUM chart in the case of a stationary first order autoregressive, AR(1) process with exponential white noise was presented by Busaba et al. (2012). Using integral equation and NIE method for CUSUM chart for ARMA(1,1) process with exponential distribution white noise, were studied focusing on analytical exact formulas of ARL_0 and ARL_1 by Phanyaem et al. (2013). Moreover, seasonal ARIMA (SARIMA) model with exponential white noise, were innovated to evaluate ARL by Paichit et al. (2014). Later, Phanyaem et al. (2014) proposed the developed NIE method for approximation of analytical ARL of autoregressive and moving average process, ARMA(p,q) process with exponential distribution white noise on EWMA and CUSUM charts and the comparison of ARL between EWMA and CUSUM charts. Recently, Petcharatet et al. (2015) derived the explicit formulas of ARLs for CUSUM chart when observations have a q order moving average, and MA(q) with exponential white noise using the integral equation. Recently, Somran et al. (2016) derived the explicit expressions and NIE method approximations of ARL for a negative CUSUM chart for a lower-sided case when observations are from the exponential distribution. The integral equation was based on Fredholm integral equations of the second kind. Later, the numerical integral equation (NIE) method of ARL on CUSUM chart for ARFIMA(p,d,q) process with exponential white noise was studied by Peerajit et al. (2016). Finally, Peerajit et al. (2016) analyzed the explicit formulas for the derivation of exact formulas from ARLs using integral equation on CUSUM chart when observations are long-memory processes with ARFIMA model in the case of exponential white noise.

In this paper, we approximate average run length (ARL) by using numerical integral equation (NIE) method base on Gauss-Legendre quadrature rule of CUSUM chart when observations are ARFIMA(p,d,q) and SARFIMA$(P,D,Q)_S$ processes with exponential white noise. The rest of this paper is organized as follows. In the next section we give a description of the ARFIMA(p,d,q) and SARFIMA$(P,D,Q)_S$ processes assumption and detail the characteristics of CUSUM chart for the ARFIMA(p,d,q) and SARFIMA$(P,D,Q)_S$ processes and ARL. The approximation of ARL by using numerical integral equation (NIE) method for long-memory process is presented in Section 3. The comparison of analytical results is shown in Section 4 and Section 5 presents conclusion and future work.

2. The Generalized Non-Seasonal and Seasonal ARFIMA Models for CUSUM Chart with Exponential White Noise

2.1. Non-seasonal ARFIMA or ARFIMA model

In time series analysis, an autoregressive fractionally integrated moving average (or fractionally integrated ARMA) model is generally denoted ARFIMA(p,d,q) where parameters p, d, and q are non-negative integers, p is the order of the autoregressive process, d is the degree of fractional differencing, and q is the order of the moving-average process proposed by Granger and Joyeux (1980) and Hosking (1981). The ARFIMA(p,d,q) is simply an extension of the ARIMA(p,d,q) notation for models, by simply allowing the degree of differencing, d, to take fractional.

Definition 1 *An ARFIMA(p,d,q) process is defined by the equation:*

$$\phi_p(B)(1-B)^d X_t = \mu + \theta_q(B)\xi_t, \tag{1}$$

where X_t *is a sequence of ARFIMA(p,d,q) process,* ξ_t *is a white noise process assumed with exponential distribution;* $\xi_t \sim Exp(\alpha)$, μ *is a constant process mean,* $(1-B)^d$ *is the fractional difference operator,* $\phi_p(B)$ *and* $\theta_q(B)$ *are autoregressive and moving-average polynomials in B of order p and q respectively given as,*

$$\phi_p(B) = (1-\phi_1 B - \phi_2 B^2 - \ldots - \phi_p B^p) \ \ and \ \ \theta_q(B) = (1-\theta_1 B - \theta_2 B^2 - \ldots - \theta_q B^q).$$

Definition 2 *A fractionally differenced white noise process, for any real-valued d is defined by:*

$$(1-B)^d X_t = \xi_t,$$

where B is the backward-shift operator, that is $B^p X_t = X_{t-p}$.

Remark 1 The fractional difference operator $(1-B)^d$; for $d \in (-0.5, 0.5)$, can be expanded by the binomial series, see Hosking (1981).

$$(1-B)^d = \sum_{p=0}^{\infty} \binom{d}{p} (-B)^p = 1 - dB - \frac{d(1-d)}{2!} B^2 - \ldots .$$

From the Definitions 1 and 2 and Remark 1 can be rearranged X_t in the generalized model.

The generalized ARFIMA(*p*,*d*,*q*) process (X_t) with exponential white noise which is used for CUSUM chart, namely:

$$\begin{aligned} X_t &= \mu + \xi_t - \theta_1 \xi_{t-1} - \theta_2 \xi_{t-2} - \ldots - \theta_q \xi_{t-q} - \left(-dX_{t-1} + \frac{d(d-1)}{2!} X_{t-2} - \ldots \right) \\ &+ \left(\phi_1 X_{t-1} - d\phi_1 X_{t-2} + \frac{d(d-1)}{2!} \phi_1 X_{t-3} - \ldots\ldots \right) + \cdots \\ &+ \left(\phi_p X_{t-p} - d\phi_p X_{t-p-1} + \frac{d(d-1)}{2!} \phi_p X_{t-p-2} - \ldots \right), \end{aligned} \quad (2)$$

where ξ_t is independent and identically distributed (i.i.d) observed sequences of exponential distribution ($\xi_t \sim Exp(\alpha)$). The initial value is normally the process mean $\xi_{t-1}, \xi_{t-2}, \ldots, \xi_{t-q} = 1$, an autoregressive coefficient $-1 \le \phi_i \le 1$; $i = 1, 2, \ldots, p$ and a moving-average coefficient $-1 \le \theta_i \le 1$; $i = 1, 2, \ldots, q$ It is assumed that the initial value of ARFIMA(*p*,*d*,*q*) process $X_{t-1}, X_{t-2}, \ldots, X_{t-p}, X_{t-(p+1)}, \ldots$ equals 1.

2.2. Seasonal ARFIMA model

For a seasonal autoregressive fractionally integrated moving average (abbreviate SARFIMA) model is straightforward extension of the ARFIMA model, proposed by Granger and Joyeux (1980) and Hosking (1981).

Definition 3 *A SARFIMA(P,D,Q)$_S$ process is defined by the equation:*

$$\Phi_P(B^S)(1-B^S)^D X_t = \mu + \Theta_Q(B^S)\xi_t, \quad (3)$$

where X_t *is a sequence of SARFIMA(P,D,Q)$_S$ process,* ξ_t *is a white noise process assumed with exponential distribution;* $\xi_t \sim Exp(\alpha)$, μ *is a constant process mean, in the practice of* $(1-B^S)^D$ *is the seasonal fractional difference operator, B is the backward-shift operator, that is,* $B^{SP}(X_t) = X_{t-SP}$, $S \in \mathbb{N}$, *S is the seasonal period (the number of time periods per year for example, S =4 with quarterly data),* $\Phi_P(B^S)$ *and* $\Theta_Q(B^S)$ *are seasonal autoregressive and seasonal moving-average in B of order P and Q respectively given as,*

$$\Phi_P(B^S) = (1 - \Phi_1 B^S - \Phi_2 B^{2S} - \ldots - \Phi_P B^{PS}) \ \ and \ \ \Theta_Q(B^S) = (1 - \Theta_1 B^S - \Theta_2 B^{2S} - \ldots - \Theta_Q B^{QS}).$$

Remark 2 The seasonal fractional difference operator $(1-B^S)^D$ with $S\in\mathbb{N}$ is seasonality for all $D\in(-0.5,\ 0.5)$ (Porter-Hudak 1990), which is expanded by the binomial series, see Hosking (1981).

$$(1-B^S)^D := \sum_{P=0}^{\infty}\binom{D}{P}(-B^S)^P = 1-DB^S-\frac{D(1-D)}{2!}B^{2S}-...,$$

Note that: In terms of $(1-B^S)^D X_t$ made a similar Definition 2, and can be rearranged X_t in the generalized model.

The generalized SARFIMA(P,D,Q)$_S$ process (X_t) with exponential white noise, namely:

$$X_t=\mu+\xi_t-\Theta_1\xi_{t-S}-\Theta_2\xi_{t-2S}-...-\Theta_Q\xi_{t-QS}-\left(-DX_{t-S}+\frac{D(D-1)}{2!}X_{t-2S}-...\right)$$
$$+\left(\Phi_1X_{t-S}-D\Phi_1X_{t-2S}+\frac{D(D-1)}{2!}\Phi_1X_{t-3S}-...\right)+\cdots$$
$$+\left(\Phi_PX_{t-PS}-D\Phi_PX_{t-(P+1)S}+\frac{D(D-1)}{2!}\Phi_PX_{t-(P+2)S}-...\right), \quad (4)$$

where $\xi_t \sim Exp(\alpha)$, the initial value $\xi_{t-S},\xi_{t-2S},...,\xi_{t-QS}=1$, a seasonal autoregressive coefficient $-1\le\Phi_i\le1$; $i=1,2,...,p$, and a seasonal moving-average coefficient $-1\le\Theta_i\le1$; $i=1,2,...,Q$. It is assumed that the initial value of SARFIMA(P,D,Q)$_S$ process $X_{t-S},X_{t-2S},...,X_{t-PS},X_{t-(P+1)S},...,$ equals 1.

As defined, X_t from (2) and (4) are stationary and invertible provided that generalized non-seasonal and seasonal ARFIMA models with both differencing parameters d and D to be non-integer and d,D are in the range $(-0.5,\ 0.5)$. If the parameters d,D are in the range $(-0.5,0)$ the processes are called to be an intermediate memory; If $d=0$ and $D=0$ the processes are short-memory (or short-range dependence) and corresponding to a standard ARMA process; finally for d,D are in the range $(0,0.5)$ the process exhibit long-memory (or long-range dependence). In this work paying particular attention to long-memory process with non-seasonal and seasonal ARFIMA models which are used for CUSUM chart is of high interest.

The upper-sided CUSUM chart is defined as the chart under the assumption $\{Y_t, t=1,2,...\}$ as a sequence of i.i.d continuous random variables with common probability density function. The CUSUM chart's statistics is expressed by the recursion:

$$Y_t=\max\{Y_{t-1}+X_t-a,\ 0\},\ \text{for } t=1,2,..., \quad (5)$$

where the chart's parameter X_t is a sequence of the generalized ARFIMA(p,d,q) and SARFIMA(P,D,Q)$_S$ processes with exponential white noise, a is positive constant called reference value, Y_0 is the starting value default by $Y_0=u,$ and u is an initial value.

The corresponding stopping time (τ_h) for the CUSUM chart is described by (5) with predetermine threshold h for can be written as:

$$\tau_h=\inf\{t>0;\ Y_t>h\},\ \text{for } u\le h, \quad (6)$$

where h is an upper control limit (UCL) of CUSUM chart.

Assume that $\{\xi_t, t = 1,2,...\}$ are i.i.d random variables with a distribution function ($F(x,\alpha)$) that are observed sequentially. Let the value of $\alpha = \alpha_0$ be the parameter before a change-point time $\theta \leq \infty$ and let the value of $\alpha > \alpha_0$ be the parameters after the change-point time.

Now, we are consider the change-point time to define more strictly ARL by denote $E_\theta(.)$ as the expectation under distribution ($F(x,\alpha)$) for a fixed change-point occurs at point θ. For $\theta = \infty$, which is a suitable chart provides large ARL and $\alpha = \alpha_0$, which is a no change-point time. Thus, the average run length for the in-control process denotes ARL$_0$, which is the expectation of stopping time (τ_h). In case of $\theta = 1$, which is small, and $\alpha_1 > \alpha_0$, they show the change-point time from α_0 to α_1. Therefore, the ARL is only evaluated with a special scenario of $\theta = 1$ of the SPC chart. Thus, the ARL for the out-of-control process denotes ARL$_1$ as follows:

$$\text{ARL} = \begin{cases} E_\infty(\tau_h) = \gamma, & \text{in-control process (ARL}_0) \\ E_1(\tau_h \mid \tau_h \geq 1), & \text{out-of-control process (ARL}_1), \end{cases} \tag{7}$$

where γ is a constant of ARL$_0$.

3. Numerical Integral Equation Method for Approximate ARL of CUSUM Chart

The concept of approximate ARL through integral equation showed that the ARL can be expressed in terms of a Fredholm integral equation of the second kind and was firstly proposed in EWMA chart by Crowder (1987). From then on, the integral equation was applied to CUSUM chart's statistic to approximate the ARL, see Champ and Rigdon (1991).

The numerical integral equation (NIE) method to compute ARL of upper-sided CUSUM chart (Y_t), let $\mathbb{P}_y$ and $\mathbb{E}_y$ be the probability measure and induced expectation corresponding to the initial value u. The function $G(u)$ is substituted the ARL of non-seasonal and seasonal ARFIMA model for CUSUM chart with initial value u, and assume the initial value of CUSUM chart's statistic $Y_0 = u; 0 \leq u \leq h.$ Then $G(u) = \mathbb{E}_u(\tau_h) < \infty$ is the solution of the integral equation.

$$G(u) = 1 + \mathbb{P}_y\{Y_1 = 0\}G(0) + \mathbb{E}_y[I\{0 < Y_1 < h\}G(Y_1)], \tag{8}$$

where Y_1 is the first observation, and $I\{0 < Y_1 < h\}$ is the indicator function, given by:

$$I\{0 < Y_1 < h\} = \begin{cases} 1 & ; 0 < Y_1 < h \\ 0 & ; \text{otherwise}. \end{cases}$$

The integral equation for $G(u)$ is derived from (8) by using Fredholm integral equation of the second kind as follows:

$$G(u) = 1 + G(0)F(a - u - X_t) + \int_0^h G(z)f(z + a - u - X_t)dz, \tag{9}$$

where $F(u) = 1 - e^{-\alpha u}$ and $f(u) = \alpha e^{-\alpha u}$.

According to the applied quadrature rule, the integral $\int_0^h f(z)dz$ can be approximated by a sum of areas of rectangles where the integral f value is chosen by base h/m with heights at the midpoints of intervals of length h/m beginning at zero. Then, with the division points $0 \leq a_1 \leq ... \leq a_m \leq h$ and weights $w_j = h/m \geq 0$ on the interval $[0,h]$, we obtain

$$\int_0^h W(z)f(z)dz \approx \sum_{j=1}^{m} w_j f(a_j) \text{ with } a_j = \frac{h}{m}\left(j-\frac{1}{2}\right), \; j=1,2,...,m, \tag{10}$$

where $W(z)$ is a weight function, a_j is a set of point and w_j is a weight define different quadrature rules.

Let $\tilde{G}(u)$ denote the approximated solution of NIE method from (9) by using the Gauss-Legendre quadrature rule as follows:

$$\tilde{G}(a_i) = 1 + \tilde{G}(a_1)F(a-a_i-X_t) + \sum_{j=1}^{m} w_j \tilde{G}(a_j) f(a_j+a-a_i-X_t), \tag{11}$$

where $i = 1,2,...,m.$

The previous equation is a system of m linear equations in the m unknowns $\tilde{G}(a_1), \tilde{G}(a_2),..., \tilde{G}(a_m),$ which can be rearranged as:

$$\begin{aligned}
\tilde{G}(a_1) &= 1 + \tilde{G}(a_1)\left[F(a-a_1-X_t) + w_1 f(a-X_t)\right] \\
&\quad + \sum_{j=2}^{m} w_j \tilde{G}(a_j) f(a_j+a-a_1-X_t) \\
\tilde{G}(a_2) &= 1 + \tilde{G}(a_1)\left[F(a-a_2-X_t) + w_1 f(a_1+a-a_2-X_t)\right] \\
&\quad + \sum_{j=2}^{m} w_j \tilde{G}(a_j) f(a_j+a-a_2-X_t) \\
&\vdots \\
\tilde{G}(a_m) &= 1 + \tilde{G}(a_1)\left[F(a-a_m-X_t) + w_1 f(a_1+a-a_m-X_t)\right] \\
&\quad + \sum_{j=2}^{m} w_j \tilde{G}(a_j) f(a_j+a-a_m-X_t)
\end{aligned}$$

Let $\mathbf{G}_{m\times 1} = \left[\tilde{G}(a_1), \tilde{G}(a_2),..., \tilde{G}(a_m)\right]'$ and $\mathbf{1}_{m\times 1} = [1,1,...,1]'$ is a column vector of $\tilde{G}(a_j)$ and one, respectively, where with, let $\mathbf{R}_{m\times m}$ is a matrix and can define the $(m,m)^{th}$ is an element of matrix $\mathbf{R}$ hereinafter as:

$$\mathbf{R}_{m\times m} = \begin{bmatrix} F(a-a_1-X_t)+w_1 f(a-X_t) \;\dots\; w_m f(a_m+a-a_1-X_t) \\ F(a-a_1-X_t)+w_1 f(a_1+a-a_2-X_t) \;\dots\; w_m f(a_m+a-a_2-X_t) \\ \vdots \qquad\qquad\qquad \vdots \\ F(a-a_m-X_t)+w_1 f(a_1+a-a_m-X_t) \;\dots\; w_m f(a_m+a-a_m-X_t) \end{bmatrix}.$$

It can be written for the matrix form as

$$\mathbf{G}_{m\times 1} = \mathbf{1}_{m\times 1} + \mathbf{R}_{m\times m}\mathbf{G}_{m\times 1}, \text{ or equivalently } (\mathbf{I}_m - \mathbf{R}_{m\times m})\mathbf{G}_{m\times 1} = \mathbf{1}_{m\times 1}, \tag{12}$$

where $\mathbf{I}_m = diag(1,1,\dots,1)$ is the unit matrix order m. If $(\mathbf{I}_m - \mathbf{R}_{m\times m})$ is invertible and exists, then approximated solution of NIE method for integral equation of matrix as follows:

$$\mathbf{G}_{m\times 1} = (\mathbf{I}_m - \mathbf{R}_{m\times m})^{-1}\mathbf{1}_{m\times 1}, \tag{13}$$

where a_j is replaced by u in $\tilde{G}(a_j)$.

The numerical integral equation (NIE) method for $\tilde{G}(u)$ of CUSUM chart can be written as

$$\tilde{G}(u) = 1 + \tilde{G}(a_1)F(a-u-X_t) + \sum_{j=1}^{m} w_j \tilde{G}(a_j) f(a_j+a-u-X_t), \tag{14}$$

with $w_j = \frac{b}{m}$, and $a_j = \frac{b}{m}\left(j - \frac{1}{2}\right)$; $j = 1, 2, \ldots, m.$

4. Comparison of Analytical Results

One of the measures of the performance of a process is the average run length (ARL). The Mathematica program is used to calculate the ARL by the simulation method. The parameters of the CUSUM chart are chosen to obtain the desired ARL_0 and at the same time minimizing the out-of-control ARL (ARL_1), for several specified shifts in the process mean, where quick detections are needed. The ARL_0 fixed at in-control ARL 370 and 500 are considered.

The ARL performance of the NIE method and the explicit formulas for the CUSUM chart in terms of percentage error will be compared and discussed when observations are ARFIMA(p,d,q) process and SARFIMA(P,D,Q)$_S$ process. The percentage error (PE) will be found by multiplying the relative error by 100%, which can be expressed as:

$$\mathrm{PE}(\%) \;=\; \frac{|\,G(u) - \tilde{G}(u)\,|}{G(u)} \times 100\%, \tag{15}$$

where $G(u)$ is derived ARL from the explicit formulas, and $\tilde{G}(u)$ is approximated ARL from NIE method.

The ARL for NIE method and explicit formulas of CUSUM chart when observations are long-memory processes with non-seasonal and seasonal ARFIMA models were computed, for shifts from the in-control process the parameter of exponential α_0, to the out-of-control process the parameter of exponential $\alpha_1 = (1+\delta)\alpha_0$. For simplicity, $\alpha_0 = 1$ and δ is equal to shift sizes of 0.01, 0.03, 0.05, 0.10, 0.20, 0.40, respectively, are presented and compared.

Table 1 ARL for the ARFIMA(3,0.35,2) using NIE method against the explicit formulas given b = 3.683115620 for ARL_0 = 370 and b = 4.0187979 for ARL_0 = 500.

a	δ	ARL_0 = 370			ARL_0 = 500		
		NIE	explicit	PE(%)	NIE	explicit	PE(%)
3.0	0.00	369.2284	370.0004	0.2086	498.8569	500.0005	0.2287
	0.01	346.1105	346.8240	0.2057	465.8002	466.8523	0.2254
	0.03	305.2639	305.8759	0.2001	407.7192	408.6136	0.2189
	0.05	270.5209	271.0485	0.1947	358.6791	359.4436	0.2127
	0.10	203.8678	204.2394	0.1819	265.6982	266.2262	0.1983
	0.20	124.3271	124.5261	0.1598	157.2251	157.4980	0.1733
	0.40	57.4286	57.5006	0.1252	69.3098	69.4032	0.1346

Table 2 ARL for the ARFIMA(3,0.35,2) using NIE method against the explicit formulas given b =3.039625 for ARL_0 = 370 and b = 3.356775 for ARL_0 = 500.

a	δ	ARL_0 = 370			ARL_0 = 500		
		NIE	explicit	PE(%)	NIE	explicit	PE(%)
3.5	0.00	369.3329	369.9999	0.1803	499.0025	500.0004	0.1996
	0.01	347.2132	347.8327	0.1781	467.5023	468.4258	0.1971
	0.03	307.9553	308.4917	0.1739	411.8734	412.6673	0.1924
	0.05	274.3666	274.8333	0.1698	364.5872	365.2732	0.1878
	0.10	209.3169	209.6529	0.1603	273.9607	274.4467	0.1771
	0.20	130.2651	130.4522	0.1434	166.0134	166.2766	0.1583
	0.40	61.7962	61.8684	0.1167	75.5292	75.6262	0.1283

Table 3 ARL for the $SARFIMA(3,0.25,1)_4$ using NIE method against the explicit formulas given b =3.908722 for ARL_0 = 370 and b = 4.255392 for ARL_0 = 500.

a	δ	ARL_0 = 370			ARL_0 = 500		
		NIE	explicit	PE(%)	NIE	explicit	PE(%)
3.0	0.00	369.2003	369.9996	0.2160	498.8184	500.0004	0.2364
	0.01	345.5876	346.3243	0.2127	464.9689	466.0528	0.2326
	0.03	303.9611	304.5898	0.2064	405.6516	406.5674	0.2253
	0.05	268.6600	269.1994	0.2004	355.7416	356.5196	0.2182
	0.10	201.2578	201.6333	0.1862	261.6354	262.1649	0.2020
	0.20	121.5514	121.7483	0.1617	153.0127	153.2796	0.1741
	0.40	55.4754	55.5443	0.1240	66.4630	66.5504	0.1313

Table 4 ARL for the $SARFIMA(3,0.25,1)_4$ using NIE method against the explicit formulas given b = 3.2189565 for ARL_0 = 370 and b = 3.539932 for ARL_0 = 500.

a	δ	ARL_0 = 370			ARL_0 = 500		
		NIE	explicit	PE(%)	NIE	explicit	PE(%)
3.5	0.00	369.3001	369.9995	0.1890	498.9579	500.0004	0.2085
	0.01	346.9550	347.6039	0.1867	467.1096	468.0732	0.2059
	0.03	307.3362	307.8969	0.1821	410.9289	411.7553	0.2007
	0.05	273.4829	273.9698	0.1777	363.2443	363.9568	0.1958
	0.10	208.0586	208.4074	0.1674	272.0718	272.5737	0.1841
	0.20	128.8751	129.0677	0.1492	163.9749	164.2439	0.1638
	0.40	60.7479	60.8211	0.1204	74.0483	74.146	0.1318

The comparison of performance between the NIE method and explicit formulas of ARL when observations are ARFIMA(3,0.35,2) and $SARFIMA(3,0.25,1)_4$ processes, are shown in Tables 1-2 and Tables 3-4, respectively. The results of ARL show differences between the two methods given by $a = 3, 3.5$, $\phi_1 = 0.10$, $\phi_2 = 0.20$, $\phi_3 = 0.30$, $\theta_1 = 0.10$, and $\theta_2 = 0.20$ for ARL_0 = 370 and 500. The NIE method used the number of division points m = 800 nods.

The in-control process ($\alpha_0 = 1$) fixes ARL_0 at 370 and 500 with shift size (δ) = 0.00. The first rows of all the tables show that the results of ARL_0 from the NIE method are close to the explicit formulas and also approach 370 and 500. The results of the ARL were computed in terms of percentage error of the NIE method and explicit formulas less than 0.25%.

It was found that if the process is out-of-control the value of exponential parameter ($\alpha_1 > \alpha_0$) presents a level shift size of 0.01, 0.03, 0.05, 0.10, 0.20, 0.40, successively. According to the results in Tables 1-4, one can see that the ARL of the NIE method and the explicit formulas are similar and tend to decrease as the level of the shift size (δ) increases. Additionally, both methods indicate that larger shifts can be detected more quickly. The results of the ARL were computed in terms of percentage error of two methods less than 0.25%. Therefore, the results of the NIE method and the explicit formulas are performance similar and in an excellent agreement in terms of the percentage error.

5. Conclusion and Future Work

This paper presented a numerical integral equation (NIE) method for approximate average run length (ARL) of CUSUM chart when observations are long-memory processes with non-seasonal and seasonal ARFIMA models in the case of exponential white noise base on Gauss-Legendre quadrature rules. From results of the aforementioned, one can see that the NIE method and explicit formulas of ARFIMA(p,d,q) and SARFIMA$(P,D,Q)_S$ processes with exponential white noise can be successfully applied to real world applications for different processes of data, for example in economics, finance, agriculture, environmental, etc. Furthermore, for a variety of data processes can be extended to other observations, such as short-memory process with non-seasonal and seasonal fractionally integrated separable spatial autoregressive (FISSAR) models and integer valued autoregressive moving-average (INARMA) models. As another future research, an extension of the proposed the NIE method's results of ARL when observations are long-memory processes with exponential white noise can be used for the application of other control charts, i.e., the EWMA chart and HWMA chart (Ramjee et al. 2000, Ramjee et al. 2002) may be studied.

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