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Partial Least Squares and Other Biased Regression Methods: A Comparative Study

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Abstract

Biased regression methods like ridge regression (RR), Liu type regression, principal component regression (PCR), partial least squares regression (PLSR) are some of the well known regression techniques which have been developed to cope with multicollinearity problem in multiple regression analysis. These help in reducing the variance of the regression parameters when the explanatory variables are multicollinear because when data is suffering from severe multicollinearity ordinary least squares (OLS) regression method leads to unstable estimates for the regression coefficients. This paper introduces a modified PLS estimator called now onwards as 'PLSLiu' estimator. The present research work compares average MSE's (AMSE) of PLSLiu with OLS, RR, PCR, Liu and PLSR estimators through simulation study. Further the work compares the MSE of both PLSR and PLSLiu estimators theoretically under the assumption of homoscedasticity. We observe that the performance of the suggested estimator is better than all the above estimators in terms of average MSE when number of explanatory variables is less than the number of units in the sample.

Keywords: Ridge regression, Liu estimator, bilinear model, principal component regression, partial least squares regression.

1. Introduction

Statisticians and researchers frequently face the problem of multicollinearity in most of the regression problems, and in such cases it is well known that the ordinary least squares regression (OLS) estimator yields poor estimates to the regression coefficients. To handle the problem of multicollinearity and non-normality, several methods have been proposed in the literature (for e.g., Hoerl and Kennard 1970, Hoerl et al. 1975, Lawless and Wang 1976, Vinod and Ullah 1981, Yeniay and Goktas 2002, Kibria 2003, Liu 2003, and El-Dereny and Rashwan 2011, etc.). Among them, the ridge regression (RR), principal component regression (PCR), Liu method and partial least squares regression (PLSR) are some of the well known methods. In recent years RR is one of the most widely used techniques in the multivariate regression analysis. It is an alternative method to the OLS that allows bias for the regression coefficients to handle the problem of multicollinearity. Liu estimation is one of the biased regression methods. Like RR, Liu estimation is also an alternative to OLS when the predictors are highly multicollinear. PCR and PLSR handle the problem of multicollinearity with few factors or components. Of these, PLSR uses much fewer factors than PCR. These two techniques are useful even when there are more variables than measurements, for e.g. Spectroscopic experiments.

2. Model

The multivariate linear regression model is

$$\boldsymbol{y} = \beta_0 + \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{u} , \tag{1}$$

where $\boldsymbol{X}$ is a $(n \times p)$ data matrix, $\boldsymbol{\beta}$ is a $(p \times 1)$ vector of regression coefficients, $\boldsymbol{y}$ is a $(n \times 1)$ vector of response, $\boldsymbol{u}$ is a ($n \times 1$) vector of residuals which are i.i.d. with zero mean and variance σ^2 and β_0 is an intercept. Suppose the data matrix $\boldsymbol{X}$ and $\boldsymbol{y}$ are mean centered, then (1) can be written as

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{u} . \tag{2}$$

If the matrix $\boldsymbol{X}$ has full rank, it is well known that ordinary least squares method gives us

$$\hat{\beta}_{\boldsymbol{OLS}} = (\boldsymbol{X'X})^{-1}\boldsymbol{X'y} . \tag{3}$$

If the data matrix $\boldsymbol{X}$ of explanatory variables suffers from severe multicollinearity, the ordinary least squares regression (OLS) estimators yield unstable estimates to the regression parameters and thus lead to poor prediction. Therefore, in order to overcome multicollinearity problem we can use some of the well known biased regression techniques like RR, PCR, PLSR, etc.

2.1. Ridge regression

When the predictors are highly correlated, the matrix $\boldsymbol{X'X}$ becomes ill-conditioned and it leads to a larger variance for the regression coefficients. To solve this problem Hoerl and Kennard (1970) proposed the ridge regression technique when the predictors are highly inter-correlated. In RR, each column of the matrix $\boldsymbol{X}$ is standardized such that $\boldsymbol{X'X}$ is in correlation form and a small positive constant k is added to every element of the diagonal of $\boldsymbol{X'X}$ which results in $\boldsymbol{X'X}$ to be non-singular. The ridge parameter (say $k \geq 0$) plays a vital role to control the bias of the regression coefficient. The ordinary ridge regression estimator is given by,

$$\hat{\beta}_{\boldsymbol{R}} = (\boldsymbol{X'X} + k\ \boldsymbol{I})^{-1}\boldsymbol{X'y}. \tag{4}$$

From (3) we write

$$\hat{\beta}_{\boldsymbol{R}} = (\boldsymbol{X'X} + k\ \boldsymbol{I})^{-1}\boldsymbol{X'X}\hat{\beta}_{\boldsymbol{OLS}}. \tag{5}$$

2.2. Liu estimator

Liu estimator was introduced by Liu (1993), which is the combination of Stein (1956) estimator and the ridge regression estimator and is defined by

$$\hat{\beta}_{\boldsymbol{Liu}} = (\boldsymbol{X'X} + \boldsymbol{I})^{-1}(\boldsymbol{XX} + d\ \boldsymbol{I})\hat{\beta}_{\boldsymbol{OLS}}. \tag{6}$$

Liu estimator is also an alternative to OLS. Liu (1993) showed that the above estimator is superior to OLS both in scalar mean squares and mean square error matrix sense. Further, this estimator is a linear function of d and thus it is easy to choose d than k in RR, as reported earlier by Alheety and Kibria (2009). The optimum value of d is represented as

$$d = 1-\sigma^2\left(\frac{\sum_{j=1}^{k-1}\frac{1}{\lambda_j(\lambda_j+1)}}{\sum_{j=1}^{k-1}\frac{\gamma_j^2}{(\lambda_j+1)^2}}\right). \tag{7}$$

The above described method does not guarantee that the estimated value of d will satisfy the condition $0<d<1$. (Akdeniz and Erol 2003, and Alheety et al. 2009). Therefore, to overcome this problem some researchers have suggested $d \in (-\infty,\infty)$ (Kaciranlar et al. 1999, Akdeniz and Erol 2003, and Alheety and Kibria 2009). However $\boldsymbol{X'X}$ is ill conditioned with a large condition number, the measure of multicollinearity, then both ridge estimator and Liu estimator can be used to estimate $\boldsymbol{\beta}$.

2.3. Bilinear regression

Bilinear modeling is one of the most widely used rank deficient techniques in applied multivariate regression analysis, for e.g. Chemometrics. Bilinear method models the spaces spanned by both $\boldsymbol{X}$ and $\boldsymbol{Y}$ block respectively. The structure of bilinear model is:

$$\boldsymbol{X} = \boldsymbol{T_m P_m'} + \boldsymbol{E_m}, \tag{8}$$

$$\boldsymbol{Y} = \boldsymbol{T_m Q_m'} + \boldsymbol{U_m}. \tag{9}$$

Here $\boldsymbol{X}$ and $\boldsymbol{Y}$ are modeled as the product of scores $\boldsymbol{T_m}$ and loadings $\boldsymbol{P_m}$ and $\boldsymbol{Q_m}$ after $m(<p)$ factors have been modeled, where $\boldsymbol{Y}$ is the matrix of order $(n\times j)$, $\boldsymbol{P_m}$ is of $(p\times m)$, $\boldsymbol{Q_m}$ is of $(j\times m)$ and $\boldsymbol{T_m}$ is of $(n\times m)$ matrices. The columns of score matrix $\boldsymbol{T_m}$ are orthogonal and the scores are some linear combinations of the original $\boldsymbol{X}$ variables. $\boldsymbol{E_m}$ and $\boldsymbol{U_m}$ are the residual matrices of $\boldsymbol{X}$ and $\boldsymbol{Y}$, respectively. The most widely used bilinear techniques are principal component regression (PCR) and partial least squares regression (PLSR). PCR is based on principal components of $\boldsymbol{X}$ variable alone whereas PLSR is based on partial least square components of both $\boldsymbol{X}$ and $\boldsymbol{Y}$ variables.

For our computational purpose we have considered $\boldsymbol{y}$ the column vector, such that $\boldsymbol{y} = \boldsymbol{T_m q_m'} + \boldsymbol{u}$.

1) Principal component regression (PCR)

PCR is also one of the widely used techniques when the matrix $\boldsymbol{X'X}$ is ill conditioned. Matrix $\boldsymbol{X'X}$ may be ill conditioned due to near or severe multicollinearity among explanatory variables. To optimize the predictive ability of the model one is more interested in obtaining the number of principal components (PCs) as they provide maximum variation of $\boldsymbol{X}$. It is an alternative method to linear regression analysis in which the response variable ($\boldsymbol{y}$) regressed on the first m PCs., which corresponds to the largest m eigen values helps to manage variance inflation.

In the case of PCR, the bilinear model after $m(< p)$ factors are taken into consideration is

$$\boldsymbol{X} = \boldsymbol{T}_m \boldsymbol{P}'_m + \boldsymbol{E}_m \quad (10)$$

$$\boldsymbol{y} = \boldsymbol{T}_m \boldsymbol{q}'_m + \boldsymbol{u}, \quad (11)$$

where $\boldsymbol{T}'_m \boldsymbol{T}_m = \boldsymbol{D}_m = \boldsymbol{diag}(\lambda_1, \lambda_2, ..., \lambda_m)$ and $\boldsymbol{P}'_m \boldsymbol{P}_m = \boldsymbol{I}$. Then the least square estimator for the loading vector $\boldsymbol{q}_m$ is given by

$$\hat{\boldsymbol{q}}'_m = (\boldsymbol{T}'_m \boldsymbol{T}_m)^{-1} \boldsymbol{T}'_m \boldsymbol{y}. \quad (12)$$

Therefore,

$$\hat{\boldsymbol{y}}_{\boldsymbol{PCR}} = \boldsymbol{T}_m (\boldsymbol{T}'_m \boldsymbol{T}_m)^{-1} \boldsymbol{T}'_m \boldsymbol{y}. \quad (13)$$

Thus PCR differs from OLS as the values of $\boldsymbol{y}$ are projected on to a subspace of the original $\boldsymbol{X}-$space. In the variable space of $\boldsymbol{X}$, the estimator of regression coefficients from PCR can be obtained by rewriting (11) as

$$\begin{aligned} \boldsymbol{y} &= \boldsymbol{X}\boldsymbol{P}_m \boldsymbol{q}'_m + \boldsymbol{u} \\ &= \boldsymbol{X}\boldsymbol{P}_m (\boldsymbol{T}'_m \boldsymbol{T}_m)^{-1} \boldsymbol{T}'_m \boldsymbol{y} + \boldsymbol{u} \\ &= \boldsymbol{X}\hat{\boldsymbol{\beta}}_{\boldsymbol{PCR}} + \boldsymbol{u} \end{aligned} \quad (14)$$

where, $\hat{\boldsymbol{\beta}}_{\boldsymbol{PCR}} = \boldsymbol{P}_m (\boldsymbol{T}'_m \boldsymbol{T}_m)^{-1} \boldsymbol{T}'_m \boldsymbol{y}$.

We write $\hat{\boldsymbol{\beta}}_{\boldsymbol{PCR}}$ as $\hat{\boldsymbol{\beta}}_{\boldsymbol{PCR}} = \boldsymbol{P}_m (\boldsymbol{T}'_m \boldsymbol{T}_m)^{-1} \boldsymbol{P}'_m \boldsymbol{X}'\boldsymbol{y}$

$$= \boldsymbol{P}_m (\boldsymbol{T}'_m \boldsymbol{T}_m)^{-1} \boldsymbol{P}'_m \boldsymbol{X}'\boldsymbol{X} \hat{\boldsymbol{\beta}}_{\boldsymbol{OLS}}. \quad (15)$$

On simplification (15) reduces to

$$\hat{\boldsymbol{\beta}}_{\boldsymbol{PCR}} = \boldsymbol{P}_m \boldsymbol{P}'_m \hat{\boldsymbol{\beta}}_{\boldsymbol{OLS}}, \quad (16)$$

where $\boldsymbol{P}_m \boldsymbol{P}'_m$ is symmetric and idempotent since $\boldsymbol{P}'_m \boldsymbol{P}_m = \boldsymbol{I}$; and since the loading vectors in PCR are orthogonal, the columns of $\boldsymbol{P}_m$ become orthonormal.

2) Partial least squares regression (PLSR)

PLS, also known as projections to latent structures, is a useful alternative to the linear multiple regression model fitted by "least squares", if the number of predictors is relatively more than the number of observations and there is more than one response variable ($\boldsymbol{y}$) and these variables are correlated. In recent years PLS is widely used in various fields, such as multivariate calibration in analytical chemistry; spectroscopy in chemometrics; and quantitative structure activity relationships (QSAR) in drug design, and so on.

The PLS method extracts orthogonal linear combinations of predictors, known as factors, from the predictor data that explain variance in both the predictor variables ($\boldsymbol{X}$) and the response variable ($\boldsymbol{y}$). To obtain the PLS estimator for $\boldsymbol{\beta}$, we write the matrix $\boldsymbol{X}$ in a bilinear form when $m(< p)$ factors are taken into consideration as

$$X = t_1 p'_1 + t_2 p'_2 + ... + t_p p'_p = \sum_{j=1}^{p} t_j p'_j = TP' \quad (17)$$

Here t_j 's are some linear combinations of $\boldsymbol{X}$ and they are orthogonal. The vector p_j is usually called as loadings. NIPALS and SIMPLS are the two popular algorithms for determining the PLS estimators. In NIPALS, orthogonality is imposed by finding t_j as linear combinations of E_j, the residual matrices. That is

$$E_j = X - \sum_{i=1}^{j} t_i p_i' \quad \text{with } E_0 = X \text{ and } t_j = E_{j-1} w_j, \tag{18}$$

where w_j is the weight vector which is orthonormal. T_j 's are calculated using w_j and then p_j 's are obtained by regressing $\boldsymbol{X}$ onto t_j.

Consider the first m dominant factors (give most of the variation in $\boldsymbol{X}$), then we write

$$\boldsymbol{T_m} = \boldsymbol{XG_m}; \quad \boldsymbol{P_m} = \boldsymbol{X'T_m}(\boldsymbol{T_m'T_m})^{-1} \text{ and } \boldsymbol{G_m} = \boldsymbol{W_m}(\boldsymbol{P_m'W_m})^{-1}. \tag{19}$$

Here, $\boldsymbol{P_m'G_m} = \boldsymbol{I_m}$ and $\boldsymbol{G_m'P_m} = \boldsymbol{I_m}$. Once the first m factors have been used for the model, the fitted $\boldsymbol{y}$ vector from PLS is

$$\hat{\boldsymbol{y}}_{\boldsymbol{PLS}} = \boldsymbol{T_m}(\boldsymbol{T_m'T_m})^{-1}\boldsymbol{T_m'y}. \tag{20}$$

Since $\boldsymbol{T_m} = \boldsymbol{XG_m}$, we have

$$\hat{\boldsymbol{y}}_{\boldsymbol{PLS}} = \boldsymbol{XG_m}(\boldsymbol{G_m'X'XG_m})^{-1}\boldsymbol{G_m'X'X}\hat{\boldsymbol{\beta}}_{\boldsymbol{OLS}}. \tag{21}$$

It is obvious that,

$$\hat{\boldsymbol{\beta}}_{\boldsymbol{PLS}} = \boldsymbol{G_m}(\boldsymbol{G_m'X'XG_m})^{-1}\boldsymbol{G_m'}(\boldsymbol{X'X})\hat{\boldsymbol{\beta}}_{\boldsymbol{OLS}}. \tag{22}$$

As in PCR, the expression for the regression coefficients in the variable space (on some steps of simplification) from PLSR is given by

$$\hat{\boldsymbol{\beta}}_{\boldsymbol{PLS}} = \boldsymbol{W_m}(\boldsymbol{P_m'W_m})^{-1}\boldsymbol{P_m'}\hat{\boldsymbol{\beta}}_{\boldsymbol{OLS}}. \tag{23}$$

Where $\boldsymbol{W_m}(\boldsymbol{P_m'W_m})^{-1}\boldsymbol{P_m'}$ is idempotent but not symmetric. Obviously,

$$E(\hat{\boldsymbol{\beta}}_{\boldsymbol{PLS}}) = \boldsymbol{G_m}(\boldsymbol{G_m'X'XG_m})^{-1}\boldsymbol{G_m'X'X\beta}, \tag{24}$$

and

$$\begin{aligned} Var(\hat{\boldsymbol{\beta}}_{\boldsymbol{PLS}}) &= E[\hat{\boldsymbol{\beta}}_{\boldsymbol{PLS}} - E(\hat{\boldsymbol{\beta}}_{\boldsymbol{PLS}})]'[\hat{\boldsymbol{\beta}}_{\boldsymbol{PLS}} - E(\hat{\boldsymbol{\beta}}_{\boldsymbol{PLS}})] \\ &= (\hat{\boldsymbol{\beta}}_{\boldsymbol{OLS}} - \boldsymbol{\beta})'\boldsymbol{X'XG_m}(\boldsymbol{G_m'X'XG_m})^{-1}\boldsymbol{G_m'G_m}(\boldsymbol{G_m'X'XG_m})^{-1}\boldsymbol{G_m'X'X}(\hat{\boldsymbol{\beta}}_{\boldsymbol{OLS}} - \boldsymbol{\beta}). \end{aligned} \tag{25}$$

Therefore, we have

$$bias(\hat{\boldsymbol{\beta}}_{\boldsymbol{PLS}}) = E(\hat{\boldsymbol{\beta}}_{\boldsymbol{PLS}}) - \beta = [\boldsymbol{G_m}(\boldsymbol{G_m'X'XG_m})^{-1}\boldsymbol{G_m'X'X} - \boldsymbol{I}]\boldsymbol{\beta}, \tag{26}$$

and

$$\begin{aligned} MSE(\hat{\boldsymbol{\beta}}_{\boldsymbol{PLS}}) &= tr(Var(\hat{\boldsymbol{\beta}}_{\boldsymbol{PLS}})) + [bias(\hat{\boldsymbol{\beta}}_{\boldsymbol{PLS}})]'[bias(\hat{\boldsymbol{\beta}}_{\boldsymbol{PLS}})] \\ &= \sum_{i=1}^{m} \frac{(x_i'x_i)^2(g_i'g_i)^2}{\lambda_i^2}(\hat{\beta}_i - \beta_i)^2 + \sum_{i=1}^{m}\left[\frac{(x_i'x_i)(g_i'g_i)}{\lambda_i} - 1\right]^2 \beta_i^2. \end{aligned} \tag{27}$$

3. Proposed Estimator

Following the method of Liu (1993) we modify the estimator defined in (22) as

$$\hat{\boldsymbol{\beta}}_{\boldsymbol{PLSLiu}} = \boldsymbol{G}_m (\boldsymbol{G}_m' \boldsymbol{X}'\boldsymbol{X}\boldsymbol{G}_m + \boldsymbol{I})^{-1} \boldsymbol{G}_m' (\boldsymbol{X}'\boldsymbol{X} + d\,\boldsymbol{I})\hat{\boldsymbol{\beta}}_{\boldsymbol{OLS}}. \quad (28)$$

The above proposed estimator performs better than all the other estimators considered in this article when sample size n is smaller than that of the number of predictors. The performance of the suggested estimator and the other estimators is evaluated under wide range of degree of multicollinearity and various error distributions.

As in (27), we write

$$MSE(\hat{\boldsymbol{\beta}}_{\boldsymbol{PLSLiu}}) = \sum_{i=1}^{m} \frac{(x_i'x_i + d)^2 (g_i'g_i)^2}{(\lambda_i + 1)^2} (\hat{\beta}_i - \beta_i)^2 + \sum_{i=1}^{m} \left[\frac{(x_i'x_i + d)(g_i'g_i)}{\lambda_i + 1} - 1 \right]^2 \beta_i^2. \quad (29)$$

Theorem: *When* $0 < d < 1$ *and* $n < p$, *the linear regression model with homoscedastic,* $\hat{\beta}_{PLSLiu}$ *is superior to* $\hat{\beta}_{PLS}$ *in the MSE sense, iff* Δ_1 *and* Δ_2 *have the same sign together, where*

$$\Delta_1 = [2x_i'x_i\lambda_i g_i'g_i + x_i'x_i g_i'g_i + d\lambda_i g_i'g_i][(\hat{\beta}_i - \beta_i)^2 + \beta_i^2] - 2\lambda_i(\lambda_i + 1)\beta_i^2$$

and $\Delta_2 = (x_i'x_i - d\lambda_i)$. (30)

Proof: To Show $\Delta = MSE(\hat{\boldsymbol{\beta}}_{\boldsymbol{PLS}}) - \text{MSE}(\hat{\boldsymbol{\beta}}_{\boldsymbol{PLSLiu}}) > 0$.

Consider,

$$MSE(\hat{\boldsymbol{\beta}}_{\boldsymbol{PLS}}) - \text{MSE}(\hat{\boldsymbol{\beta}}_{\boldsymbol{PLSLiu}})$$

$$= \sum_{i=1}^{m} \left\{ \frac{(x_i'x_i)^2}{\lambda_i^2} - \frac{(x_i'x_i + d)^2}{(\lambda_i + 1)^2} \right\} (g_i'g_i)^2 (\hat{\beta}_i - \beta_i)^2 + \sum_{i=1}^{m} \left\{ \left[\frac{(x_i'x_i)(g_i'g_i)}{\lambda_i} - 1 \right]^2 - \left[\frac{(x_i'x_i + d)(g_i'g_i)}{\lambda_i + 1} - 1 \right]^2 \right\} \beta_i^2$$

$$= \sum_{i=1}^{m} \left\{ \frac{(2x_i'x_i\lambda_i + x_i'x_i + d\lambda_i)(x_i'x_i - d\lambda_i)}{\lambda_i^2(\lambda_i + 1)^2} \right\} (g_i'g_i)^2 (\hat{\beta}_i - \beta_i)^2 + \sum_{i=1}^{m} \left\{ \left[\frac{(2x_i'x_i g_i'g_i\lambda_i + x_i'x_i g_i'g_i + d\lambda_i g_i'g_i)}{\lambda_i(\lambda_i + 1)} - 2 \right] \left[\frac{(x_i'x_i - d\lambda_i) g_i'g_i}{\lambda_i(\lambda_i + 1)} \right] \right\} \beta_i^2.$$

(31)

On simplification, we have

$$MSE(\hat{\boldsymbol{\beta}}_{\boldsymbol{PLS}}) - \text{MSE}(\hat{\boldsymbol{\beta}}_{\boldsymbol{PLSLiu}})$$

$$= \sum_{i=1}^{m} \left\{ \begin{array}{l} [2x_i'x_i\lambda_i g_i'g_i + x_i'x_i g_i'g_i + d\lambda_i g_i'g_i](\hat{\beta}_i - \beta_i)^2 + \\ \quad [2x_i'x_i g_i'g_i\lambda_i + x_i'x_i g_i'g_i + d\lambda_i g_i'g_i) - 2\lambda_i(\lambda_i + 1)]\beta_i^2 \end{array} \right\} g_i'g_i \frac{(x_i'x_i - d\lambda_i)}{\lambda_i^2(\lambda_i + 1)^2}$$

$$= \sum_{i=1}^{m} \left\{ [2x_i'x_i\lambda_i g_i'g_i + x_i'x_i g_i'g_i + d\lambda_i g_i'g_i][(\hat{\beta}_i - \beta_i)^2 + \beta_i^2] - 2\lambda_i(\lambda_i + 1)\beta_i^2 \right\} g_i'g_i \cdot \frac{(x_i'x_i - d\lambda_i)}{\lambda_i^2(\lambda_i + 1)^2}. \quad (32)$$

Thus, for $0 < d < 1$,

$$\Delta = \sum_{i=1}^{m} \Delta_1 \Delta_2 \left(\frac{g_i'g_i}{\lambda_i^2(\lambda_i + 1)^2} \right) > 0, \quad (33)$$

Provided Δ_1 and Δ_2 have the same sign together, where Δ_1 and Δ_2 are defined in (30).

4. Simulation Study

The performance of RR, Liu, PCR, PLS, and PLSLiu estimators are studied through Monte Carlo simulation. Here we have investigated the performance of these estimators in the presence of a low, moderate, and high degree of multicollinearity. The results are obtained by generating a random matrix $\boldsymbol{X}$ of size ($n \times p$), where $p = 25$, using the relation:

$$x_{ij} = (1-\rho^2)^{1/2}\xi_{ij} + \rho\xi_{ip}, i = 1, 2, ..., n;\ j = 1, 2, ..., p,$$

where ξ_{ij} is an independent standard normal pseudo-random number, ρ is specified such that ρ^2 is the correlation between any two predictors. These predictor variables are standardized such that $\boldsymbol{X'X}$ is in the correlation form and it is used to generate $\boldsymbol{y}$ with $\boldsymbol{\beta} = ($ 4, 3, 2.3, 1, 3.07, 4, 3, 2, 1.4, 3, 4, 3, 2.11, 1, 3, 4, 3, 2.03, 1, 3, 4.05, 3, 2, 1.1, 3 $)'$. To study the performance of different estimators we have assumed that error follows $N(0, \sigma^2)$ as the first case. The simulation is carried out for various values for n: 5, 10, 15 and 20; and error variances (σ^2) are 5, 10, 25, 50, 100 and 1000; and ρ as 0.2, 0.4, 0.7, 0.9, 0.99, and 0.9999. The Experiment was repeated 1000 times each and the average mean square error (AMSE) was computed. In the process of simulation we have considered the minimum of AMSE of PLSLiu against AMSE of other biased regression methods to be the best in terms of MSE point of view.

Ridge and Liu estimates are computed for p-components of $\boldsymbol{X}$. Whereas, PCR, PLS and the suggested estimator PLSLiu estimates are computed for first $m(< p)$ dominant factors, and these factors capture most of the variance in $\boldsymbol{X}$ where $\boldsymbol{X}$ takes the bilinear form so that $\boldsymbol{X}$ is expressed as a linear combination of the product of scores and loadings. Tables 1, 2, and 3 indicated that the performance of the suggested estimator (PLSLiu) is better than all the other estimators in almost all the cases.

To verify the robustness of the proposed estimator (PLSLiu) we have considered the non-normal distributions for the error term also. In this section, we studied the performance of the estimators in terms of AMSE as the first case when error follows t-distribution with 5 d.f., and similarly as the second case we have considered the standard Cauchy distribution for the error term. Here also the experiment was repeated for 1000 times each for the same set of values of n, ρ and $\boldsymbol{\beta}$, and in both the cases, the suggested estimator (PLSLiu) show better performance than all the other estimators considered under study. However, if we observe carefully when n, approaches p the differences in AMSE of PLS and PLSLiu become smaller; and in this case, PLS dominates PLSLiu in most of the cases as suggested by the number of frequencies ($\Delta > 0$), which were obtained out of 1,000 iterations,through simulation study. Since the performance of the suggested estimator and the other estimators is verified under a wide range (i.e., a low, moderate, and high) of degrees of collinearity (ρ), sample size (n) and different error distribution ($N(0, \sigma^2)$; $t_{(5)}$ d.f. and Cauchy) for the error term, we may conclude that the performance of the proposed estimator is satisfactory and comparable; and further one can think about the usefulness of the proposed estimator in various industries.

Table 1 Average of mean square error (AMSE-in terms of 10^4) of different estimators when error follows $N(0, \sigma^2 I)$

n	ρ	σ^2	$AMSE(\hat{\beta}_{OLS})$	$AMSE(\hat{\beta}_{R})$	$AMSE(\hat{\beta}_{Liu})$	$AMSE(\hat{\beta}_{PCR})$
5	0.2	5	1516.5304	1572.2501	1246.2122	81.0607
		10	223.3789	216.4736	186.0384	22.0088
		25	8576.9938	7641.5067	7251.1269	1185.6709
		50	19824.9900	19654.6112	16436.1343	1597.1447
		100	136833.8401	120915.6429	115841.3954	16404.7644
		1000	61992620.2257	61468926.4688	51385298.9891	2511785.5830
	0.4	5	8857.9006	8381.7805	7409.8663	1049.5929
		10	37398.4823	36672.1594	31070.9125	2494.4294
		25	3229.3509	3165.7491	2683.0799	208.1658
		50	31166.1095	30543.7819	25891.9096	2057.5966
		100	20163.5678	20391.6638	16651.1585	1249.7192
		1000	3604335.4169	3703759.4873	2966923.1430	165144.2671
	0.7	5	1735.1961	1728.2748	1437.4558	133.1294
		10	2852.2268	2660.8199	2392.1235	387.4315
		25	15540.5835	15888.0683	12808.1052	1169.6524
		50	17617.3134	15988.5404	14849.4446	2525.3681
		100	19767.3558	20458.7210	16249.0157	1102.6470
		1000	5646376.5287	5890840.3261	4633306.7762	272341.0484
	0.9	5	329.0574	323.1454	273.3191	25.7092
		10	90445.6219	87489.7374	75322.2923	4408.5894
		25	114791.5275	122900.3935	93692.4575	3812.6519
		50	7737.3456	7465.1201	6450.1915	811.0184
		100	7377.9546	7306.1960	6119.7483	696.8628
		1000	14025286.1459	12846996.2256	11798803.4196	1542676.3939
	0.99	5	1009.2056	1016.3978	834.1964	83.8264
		10	561333.2800	573666.2759	462515.1338	5816.3485
		25	47833.7719	47511.6921	39651.5826	4812.2068
		50	4864.4586	4923.2990	4017.0043	358.5192
		100	1605806.5913	1399388.4845	1361859.4557	209250.0749
		1000	2143563.4228	2075242.9859	1785302.4584	184394.0943
	0.9999	5	4963.0491	5124.3967	4081.5819	270.8775
		10	826.6442	847.7868	680.7751	53.0380
		25	5266.1310	5004.9009	4402.0315	660.2419
		50	3512.9656	2909.8507	3007.5669	847.8697
		100	5965.9394	5866.3897	4954.5414	526.5364
		1000	16527736.5025	16761695.8569	13640345.4858	926494.1665

Table 1 (Continued)

n	ρ	σ^2	$AMSE(\hat{\beta}_{PLS})$	$AMSE(\hat{\beta}_{PLSLiu})$	$\Delta > 0$
5	0.2	5	3.5510	0.9834	917
		10	1.1855	0.3492	930
		25	63.7942	17.4820	964
		50	86.2945	23.5439	989
		100	1050.6355	287.4672	997
		1000	259957.8487	71852.8167	1000
	0.4	5	51.9351	14.0927	926
		10	110.4845	30.1629	910
		25	15.3275	4.2230	961
		50	148.1856	40.9650	985
		100	70.9729	19.4141	996
		1000	10627.9251	2898.3791	1000
	0.7	5	6.6999	1.8364	917
		10	13.9665	3.8192	936
		25	48.1197	12.8597	956
		50	107.2646	29.1188	991
		100	53.3576	14.4118	999
		1000	12957.0350	3531.0669	1000
	0.9	5	1.4979	0.4365	932
		10	473.7792	131.3766	929
		25	141.4741	37.5891	966
		50	41.3352	11.0898	987
		100	30.9997	8.3068	997
		1000	97212.9663	26743.0990	1000
	0.99	5	3.8029	1.0532	920
		10	1669.5457	463.1582	934
		25	198.4648	53.0606	960
		50	15.7111	4.2538	988
		100	14300.5355	4023.2925	999
		1000	11187.9776	3051.4319	1000
	0.9999	5	13.6123	3.7102	924
		10	2.5031	0.7054	934
		25	29.8591	8.0675	943
		50	32.6919	8.7949	982
		100	26.5531	7.2083	995
		1000	51707.4632	14156.1188	1000

Table 1 (Continued)

n	ρ	σ^2	$AMSE(\hat{\beta}_{OLS})$	$AMSE(\hat{\beta}_{R})$	$AMSE(\hat{\beta}_{Liu})$	$AMSE(\hat{\beta}_{PCR})$
10	0.2	5	4153120.7771	2157774.5336	3766730.3024	1858693.3652
		10	27300.5043	28580.1685	22387.0379	1412.2370
		25	7860.1562	7485.3097	6564.0466	841.0871
		50	92187.7446	78411.6898	78518.2233	18483.4843
		100	32011.3706	33830.4128	26192.1797	924.0165
		1000	554686.2082	528096.4169	463323.9640	55448.2961
	0.4	5	304.3960	276.1100	256.5583	46.2724
		10	170.7118	147.4745	145.0569	29.3610
		25	1069.9092	940.8237	906.0417	173.4289
		50	7781.1952	7137.2377	6540.7137	768.6375
		100	31408.7991	27112.7980	26683.9694	4810.9834
		1000	1134958.0219	1028288.3864	956537.5459	147616.4204
	0.7	5	1150.8776	1007.3687	976.3193	210.5459
		10	3516.3074	2982.2682	2995.6416	610.3928
		25	16575.1542	17140.9946	13620.2765	464.5369
		50	10646.2659	9307.0746	9025.5703	1612.0038
		100	48044157.9418	50778220.7063	39319538.1972	1925800.6114
		1000	1241553.6959	1146491.6673	1042844.5958	158859.3695
	0.9	5	50304.1303	53242.3146	41155.4075	22056911
		10	267.1426	191.3070	233.6488	859526
		25	20486.4605	9205.9008	18814.4543	116014495
		50	11933.4286	9531.5647	10246.8918	17463783
		100	268752.4434	218613.8974	230696.2583	658505624
		1000	466904.3340	434442.6218	391631.6674	555562778
	0.99	5	1670.6843	1410.3113	1424.5976	291.0426
		10	55428.8422	56019.7685	45779.4910	3982.6823
		25	122810.4936	114014.4559	103100.0321	17323.7636
		50	2131.4592	1638.5390	1842.4148	398.7634
		100	3721.0111	3166.3559	3168.3435	697.3921
		1000	814718.9774	650553.5470	700966.3374	179650.4316
	0.9999	5	11334.2128	11376.2212	9373.6400	643.8633
		10	5557.7103	3737.3116	4894.7244	1668.9977
		25	1575.2803	1398.5597	1332.1595	267.0096
		50	19632.8915	18560.8013	16407.0481	1910.8561
		100	9523.4949	8953.7659	7972.6520	1085.2292
		1000	559390280.0557	543704806.3903	465015123.0861	17771009.9152

Table 1 (Continued)

n	ρ	σ^2	$AMSE(\hat{\beta}_{PLS})$	$AMSE(\hat{\beta}_{PLSLiu})$	$\Delta > 0$
10	0.2	5	90950.9527	24746.1648	866
		10	62.4174	17.0452	886
		25	40.9012	11.5173	967
		50	760.5264	212.3478	994
		100	57.3179	16.2787	1000
		1000	3185.3291	885.6549	1000
	0.4	5	2.1697	0.6189	901
		10	1.5017	4.3989	913
		25	8.9862	2.5623	959
		50	54.4775	15.5106	998
		100	283.9258	79.5088	1000
		1000	7841.7032	2171.8602	1000
	0.7	5	9.6067	2.6234	884
		10	32.9742	9.3434	909
		25	45.4628	13.2244	957
		50	93.5208	26.3336	998
		100	92520.7420	25073.4255	1000
		1000	8175.4549	2272.3985	1000
	0.9	5	92.3893	25.1022	884
		10	3.9016	1.0738	899
		25	487.2792	131.5395	974
		50	138.3109	41.3402	995
		100	2963.9638	800.2536	1000
		1000	2987.4520	831.8690	1000
	0.99	5	16.0761	4.5382	847
		10	188.1295	51.2022	911
		25	822.3595	222.3016	973
		50	26.4527	7.5584	997
		100	29.7638	8.4504	999
		1000	8228.2906	2274.4813	1000
	0.9999	5	43.9977	12.0971	881
		10	90.6187	25.1343	903
		25	12.7803	3.5903	955
		50	113.5106	33.6741	998
		100	58.4711	16.3716	999
		1000	2778207.3496	830655.9804	1000

Table 1 (Continued)

n	ρ	σ^2	$AMSE(\hat{\beta}_{OLS})$	$AMSE(\hat{\beta}_{R})$	$AMSE(\hat{\beta}_{Liu})$	$AMSE(\hat{\beta}_{PCR})$
15	0.2	5	277.9911	227.9873	237.7975	48.1675
		10	308.2518	249.1521	264.5394	66.0390
		25	1151.8209	1177.0779	948.4464	56.9676
		50	5255.8313	4295.7590	4493.6213	779.0259
		100	7651.1623	6697.0881	6477.4180	1125.5228
		1000	7179695466.6168	7187276067.5398	5936103140.7670	408244834.6121
	0.4	5	415.1040	316.5218	359.0923	92.1635
		10	503.6135	404.2725	431.7028	80.6990
		25	12120.2201	11316.8598	10152.5513	972.6687
		50	60747228.5260	40211607.7703	53699900.4559	23119501.9875
		100	62182.9836	45865.0545	53938.6584	14923.7990
		1000	1139894.0047	1013340.7622	962317.8765	142499.4198
	0.7	5	6173751.0301	5708101.8705	5180140.0635	688425.7955
		10	775.0758	673.5462	656.6839	76.4164
		25	2128.1796	1726.2830	1824.1064	351.1241
		50	722755.2858	632737.4364	612065.7374	105564.2353
		100	17070.6072	10059.0128	15203.9907	2166.5789
		1000	3001957.6618	3093425.2186	2468042.2247	106345.1408
	0.9	5	166.2385	131.7192	142.9917	31.7288
		10	353.0526	260.5000	307.0060	93.2799
		25	572.5787	500.6015	485.1825	95.4259
		50	1401.3105	1284.0016	1177.2623	157.5716
		100	13812.8299	12847.5553	11577.8826	1563.6370
		1000	848475.1256	755408.0455	716748.0841	124681.4584
	0.99	5	1417.2756	1075.4346	1225.8958	303.8681
		10	2593.9191	1316.3532	2345.3942	591.7267
		25	1096.3611	708.8287	969.8271	391.0106
		50	11002.6010	7651.7817	9626.9074	1829.1399
		100	361418.9822	227990.7939	319561.5835	82086.8275
		1000	994570.5535	953263.1810	828978.2831	85447.3688
	0.9999	5	768232.0628	778460.2989	633874.8804	49368.8779
		10	875.8925	718.8876	749.1686	164.3856
		25	18447.3345	11501.7550	16400.8976	7628.0962
		50	4398.1887	4047.3890	3692.9734	543.3182
		100	211976.5246	200120.5066	177243.6511	24175.0260
		1000	4357973.5096	3699998.9994	3707024.5234	481108.0239

Table 1 (Continued)

n	ρ	σ^2	$AMSE(\hat{\beta}_{PLS})$	$AMSE(\hat{\beta}_{PLSLiu})$	$\Delta > 0$
15	0.2	5	2.9132	0.9065	741
		10	3.3602	0.9906	774
		25	3.7002	1.2031	945
		50	39.1468	12.9830	997
		100	64.1207	19.7799	1000
		1000	29384896.1877	9205836.5053	1000
	0.4	5	4.7312	1.4446.	718
		10	5.6877	1.8329	800
		25	78.7804	23.2903	936
		50	991222.5440	265427.3220	996
		100	864.2333	298.3810	1000
		1000	9314.6396	3139.0163	1000
	0.7	5	42857.9911	12528.0143	773
		10	6.7810	2.1116	792
		25	23.3607	6.8943	949
		50	6225.8600	1832.7610	996
		100	334.9948	103.6098	1000
		1000	8497.3728	2622.3991	1000
	0.9	5	19344	0.6040	744
		10	49170	1.4211	796
		25	49588	1.4629	953
		50	99566	3.0937	995
		100	917814	27.5663	1000
		1000	66870356	1960.4875	1000
	0.99	5	18.6243	5.6921	750
		10	58.1904	17.6057	791
		25	15.2304	4.2587	944
		50	165.0759	49.6730	998
		100	6389.7769	1969.5006	1000
		1000	5327.8608	1623.3658	1000
	0.9999	5	1680.5230	465.2219	737
		10	9.0619	2.8083	769
		25	335.9434	92.3312	941
		50	30.7856	9.6995	999
		100	1310.9511	385.0238	1000
		1000	41729.9779	12600.1527	999

Table 1 (Continued)

n	ρ	σ^2	$AMSE(\hat{\beta}_{OLS})$	$AMSE(\hat{\beta}_{R})$	$AMSE(\hat{\beta}_{Liu})$	$AMSE(\hat{\beta}_{PCR})$
20	0.2	5	32905.1953	31058.8578	27475.3378	2229.8897
		10	2534.1197	2428.6661	2110.2997	208.5113
		25	51333.1887	26086.5839	46099.7318	13542.3044
		50	990.9591	790.4722	848.9566	157.2528
		100	19228.0446	12617.9247	16936.3774	4608.0815
		1000	1661474.2541	1068095.2343	1462857.8807	382877.5660
	0.4	5	52.3432	36.6402	45.7216	12.6773
		10	787.2977	644.9679	672.2509	126.3778
		25	8089.3077	8320.7106	6651.9604	353.4188
		50	786.7524	577.4148	682.2431	139.1573
		100	133193.0369	58086.3635	121278.1708	32572.7379
		1000	1026470.7288	1069979.6645	841744.9763	37549.5513
	0.7	5	274.4378	190.0067	240.1719	75.9530
		10	80733.9204	61724.8471	69659.4047	16389.4377
		25	290.1160	273.5969	242.4275	27.4205
		50	17131.4095	11823.8553	15017.9025	4881.5440
		100	171624.4382	182558.8753	140100.0142	3151.9813
		1000	948324.5040	491914.5324	852100.1783	336414.5555
	0.9	5	63846.1143	54117.7697	54224.8003	6024.5695
		10	279.9929	203.8115	242.9710	51.7370
		25	2175.2990	2136.6557	1804.0389	149.1396
		50	305045.4065	323081.7683	249320.2802	8625.8495
		100	38464.2952	40312.0757	31509.4916	1081.9500
		1000	3478952.9298	3743742.2846	2834719.1235	56064.9288
	0.99	5	71854.1840	19341.4962	67861.9291	47312.2055
		10	6586.5399	5424.9533	5626.4356	875.8981
		25	89099.1625	9992.3566	85902.7059	62222.9071
		50	17787.5133	15649.8659	15040.7233	2568.6032
		100	5262.8820	4733.0614	4435.5389	699.0727
		1000	17344593.9274	17073784.5680	14382849.5501	898644.7596
	0.9999	5	931.3029	926.3231	770.5056	58.2641
		10	206.1989	133.5302	181.3257	45.4724
		25	8821.5616	8906.8412	7281.2120	449.6697
		50	4795.2184	4537.9923	4000.1544	317.9368
		100	6861575.5973	3342178.0689	6221172.0813	2377501.9142
		1000	1372379.6137	1398789.0160	1130175.3413	56249.3280

Table 1 (Continued)

n	ρ	σ^2	$AMSE(\hat{\beta}_{PLS})$	$AMSE(\hat{\beta}_{PLSLiu})$	$\Delta > 0$
20	0.2	5	191.4858	77.5080	472
		10	14.3905	6.4122	504
		25	1238.7065	871.9234	737
		50	12.2608	7.2286	897
		100	334.0107	112.4984	896
		1000	29326.1917	10138.7316	893
	0.4	5	0.8058	0.3510	424
		10	8.0934	4.3554	497
		25	24.3707	8.3201	715
		50	11.2778	4.4169	867
		100	3309.6068	1181.9484	908
		1000	2471.5171	986.9668	898
	0.7	5	4.3633	1.6796	461
		10	1021.1745	367.2518	487
		25	1.7889	0.7275	742
		50	268.7698	82.8166	891
		100	326.6070	239.9570	898
		1000	23293.4353	14994.6413	882
	0.9	5	632.3587	226.7970	452
		10	3.8608	1.4446	509
		25	10.1878	4.4423	723
		50	596.8745	256.9225	887
		100	86.1990	32.0469	887
		1000	4337.3397	1710.9765	910
	0.99	5	2142.9259	611.9920	452
		10	68.4663	24.5348	514
		25	3251.9595	1227.7711	711
		50	163.3076	109.4603	878
		100	41.0995	14.1020	902
		1000	81955.4354	25843.1482	907
	0.9999	5	3.8820	1.6764	452
		10	3.6445	1.5457	517
		25	33.6296	16.0349	725
		50	29.3274	15.6739	892
		100	158857.1315	50129.1208	891
		1000	4479.0396	1819.9875	896

Table 2 Average of mean square error (AMSE – in terms of 10^4) of different estimators when error follows *t*-distribution with 5 d.f.

n	ρ	$AMSE(\hat{\beta}_{OLS})$	$AMSE(\hat{\beta}_{R})$	$AMSE(\hat{\beta}_{Liu})$	$AMSE(\hat{\beta}_{PCR})$	$AMSE(\hat{\beta}_{PLS})$	$AMSE(\hat{\beta}_{PLSLiu})$	$\Delta > 0$
5	0.2	87157.0512	81329.0797	73072.3215	6216.5455	562.1903	155.1620	912
	0.4	69981.2215	67052.8785	58369.6244	1782.8222	392.8284	110.7337	923
	0.7	2215.4169	2073.3389	1857.1497	324.6737	12.0146	3.2711	934
	0.9	1418.5700	1447.0561	1169.5761	89.5241	4.4798	1.2341	928
	0.99	424.0192	404.1341	354.2257	49.7636	2.3075	0.6545	921
	0.9999	127498.6081	131541.1892	104868.1717	6299.2744	350.1245	95.1719	922
10	0.2	12255.8127	9934.3011	10520.3430	2455.5123	134.6949	37.1662	864
	0.4	1499.9045	983.0251	1325.2152	466.0922	24.8417	6.9309	845
	0.7	424.9516	389.2038	357.3798	50.3488	3.0059	0.8700	873
	0.9	1270.0762	1206.5472	1061.2389	127.2345	7.4410	2.1054	851
	0.99	13449.4314	10569.7920	11604.7400	2886.8668	146.9787	39.9608	857
	0.9999	1953091.7213	2019892.3719	1605488.3591	102292.2597	5274.2211	1448.4606	879
15	0.2	1895.3955	1755.0897	1588.7218	226.9285	12.1439	4.0157	734
	0.4	1058.9558	744.5604	923.3153	179.7136	16.3069	5.3483	722
	0.7	3075.3336	2743.7536	2597.8491	403.1130	20.8655	5.9176	725
	0.9	1270.9712	828.2925	1119.2132	296.6323	19.8971	6.2872	757
	0.99	379.2210	340.8718	319.7388	50.5859	2.8515	0.8791	693
	0.9999	15398.0697	11443.6294	13352.4437	3863.5671	173.2534	53.0771	736
20	0.2	525.7875	349.6401	462.1781	132.3481	8.9200	3.2840	450
	0.4	613.1243	415.6343	537.0500	122.6883	9.8156	4.0286	448
	0.7	379.3106	322.3493	322.4829	54.5353	3.6242	1.6038	458
	0.9	208.3873	164.3533	179.1123	36.0422	2.4630	0.8930	456
	0.99	597.4312	126.3627	569.6564	347.6495	18.3274	5.5953	452
	0.9999	50.4408	42.1766	42.9233	7.4760	0.5340	0.3123	444

Table 3 Average of mean square error (AMSE – in terms of 10^4) of different estimators when error follows Cauchy distribution

n	ρ	$AMSE(\hat{\beta}_{OLS})$	$AMSE(\hat{\beta}_{R})$	$AMSE(\hat{\beta}_{Liu})$	$AMSE(\hat{\beta}_{PCR})$	$AMSE(\hat{\beta}_{PLS})$	$AMSE(\hat{\beta}_{PLSLiu})$	$\Delta > 0$
5	0.2	9317.1480	8382.3819	7864.5407	884.3745	64.4393	17.5496	941
	0.4	622.3704	594.3602	519.7731	70.8307	3.0564	0.8543	939
	0.7	24644.7850	25875.5289	20197.9560	1004.8048	52.7798	14.2330	919
	0.9	17293.4755	18091.6572	14183.0160	527.8911	38.4545	10.4716	918
	0.99	32811.3958	31602.1367	27354.3210	3445.8812	152.5327	41.6468	924
	0.9999	3614.4219	3531.8403	3005.0369	240.4617	17.4639	4.7931	931
10	0.2	1523.9098	1365.4060	1286.8262	232.5602	11.6450	3.2658	886
	0.4	4390.7389	3532.9021	3774.4726	1074.2859	49.8537	13.6236	910
	0.7	12746.1829	10514.6772	10896.8951	1580.8035	136.2253	39.5889	887
	0.9	2110731.6339	2117111.9701	1745906.3954	175446.6869	7936.6307	2168.8611	858
	0.99	398.8358	346.0762	338.3899	52.8062	3.4856	1.0222	898
	0.9999	4960.2901	4750.9314	4139.3819	577.4284	27.2923	7.5292	901
15	0.2	1923.4286	1774.4184	1613.8606	214.1564	13.1139	4.0637	795
	0.4	14727762.8712	15134045.2363	12110784.9043	198701.8323	3847.76685	12294.0563	762
	0.7	1399.0676	1000.7062	1216.2675	211.6469	20.6449	6.9799	785
	0.9	9432.1206	7059.3538	8180.4404	2389.4579	106.2658	30.7650	786
	0.99	611933.6091	471628.7206	527166.8365	85376.6471	7907.1269	2605.2298	767
	0.9999	962.1307	823.8597	816.9093	96.1428	8.9122	2.7886	765
20	0.2	17918.1583	16309.7760	15059.1736	2036.5102	129.3614	48.3618	529
	0.4	4931.5762	2635.7229	4440.7469	1893.5045	108.1693	39.7737	554
	0.7	873.8902	805.2176	732.6926	89.6886	6.2640	2.6053	535
	0.9	395.2861	350.8500	334.0235	58.2496	3.3420	1.2087	491
	0.99	43725.3648	42984.5049	36240.2650	2386.8416	195.3115	82.5914	528
	0.9999	10528.6356	8686.3383	8998.1807	1651.9909	107.6163	34.4691	569

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